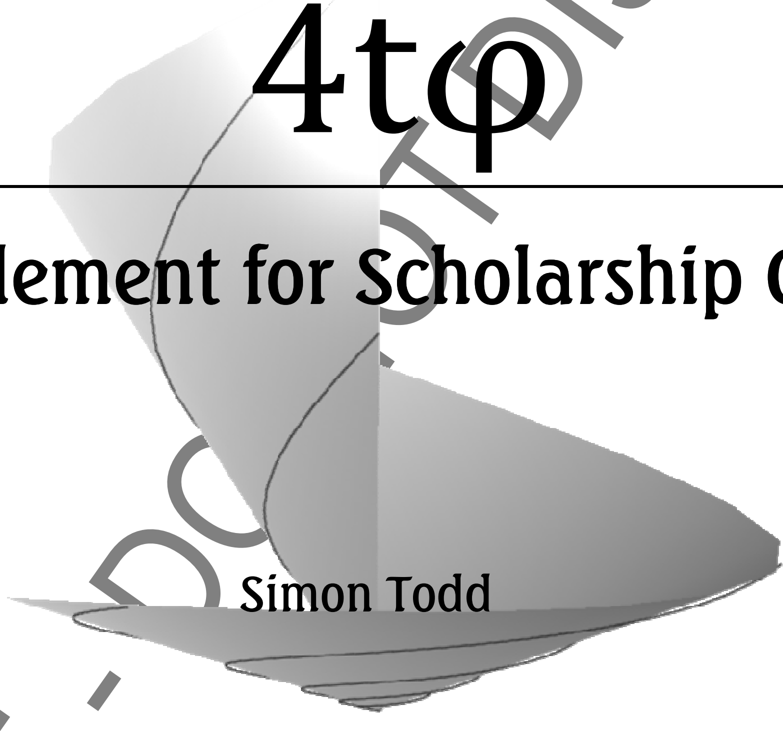

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A supplement for Scholarship Calculus

Simon Todd

YEAR 12 + 13

TEACHER'S NOTES

With thanks to Kevin Quinn, Dean Mckenzie
and the group of students from St. Andrew's College who were my guinea pigs.

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**It is expected that any student being taught from this material
owns their own copy of the relevant Student Workbook.**

FOREWORD

The author, Simon Todd, is, in my opinion, a brilliant mathematician. In the four years I have known him, I have been impressed with his professionalism and the way that he learnt very early that a great deal of hard work, combined with his exceptional mathematical ability, would be an unbeatable combination. This resulted in his elevation to Dux of St. Andrew's College and, more importantly, gaining an Outstanding Scholarship in Calculus. Since leaving St. Andrew's College he has continued to excel, winning a myriad of scholarships.

I have worked closely with Simon as he has developed his programme for Scholarship students over the last two years at St. Andrew's College. The course is set to a high standard and the author makes no apology for the workload expected of the students on their journey of discovery.

To the teachers guiding the studies of their students, I can unequivocally recommend the textbooks supplied for the course, provided those who use them have the ability to fully appreciate their contents.
Since algebraic skills are such an important part of a student's armoury, a large focus of the Year 12 workbook is to develop and reinforce these skills before introducing the study of conic sections. The Year 13 workbook then proceeds to extend the NCEA Level Three curriculum and introduce higher-level techniques which will be useful at Scholarship and necessary for tertiary study.

A great facet of the workbooks is the number of problems which must be solved in a general way before encountering numerical examples, leading, we hope, to a comfortable alliance with abstract thinking. While some general derivations and extensions are absent from the Student Workbooks due to the potential difficulty that students not at the Outstanding level may have with them, they have all been included in the Teacher's Notes. We believe that fully establishing these by guiding students through them on the board is an invaluable exercise for building comprehension and mathematical appreciation and will encourage students to emulate such abstract thinking in the Scholarship examination and in later studies.

To the students, meeting work at this level for the first time, I offer encouragement. NCEA has probably been a doodle in the park for you, so being abruptly forced to face the reality of high level mathematical thinking for the first time will be a sobering lesson. If you come through this phase with your enthusiasm intact and your mathematical skills heightened, you will have earned your place in this advanced programme.

Finally, the course is difficult but exciting. As you progress, you will discover that there is no single way to solve a problem. Some methods are long-winded and pedestrian, but with flexibility of mind, you may end up with an elegant solution admired by all. Good luck.

Kevin Quinn, December 2010.

NOTE FROM THE AUTHOR

This package has been designed as a supplement for students who are or will be candidates for Scholarship Calculus. The package is intended to support a 2-year course beginning in Year 12 which requires an extra one hour of class time per week for the first three terms of each year.

There are four parts to this package: the Teacher's Notes (this book), which contains the student material for both year levels plus extra notes offering extension aimed at Outstanding Scholarship candidates; two Student Workbooks (one at each level), which contain notes and exercises with space to write answers; and the Answer Booklet, which contains the complete worked answers to every question asked in each of the texts.

In the Year 12 part of the course, students will work primarily on aspects of real algebra and conic sections. These areas include many parts which are foundational to many different types of questions and to further studies, and it is therefore important that students gain competence in these areas to allow them to handle more complex areas with ease. In the Year 13 part of the course, students will cover the remaining aspects of the NCEA Level 3 curriculum – differentiation, trigonometry, integration and complex numbers. The Year 13 material also includes 16 Scholarship-style Homework Assignments, intended to be set once a fortnight to give students practice at the higher, extended levels of thinking required for the exam.

As is appropriate for a supplementary extension text, not every part of the curriculum is included in this package: some of the basics have been left out. If this package is taught alongside normal NCEA Level Three classes, this should not be an issue. On the other hand, the package does include many extensions beyond the level of the curriculum. The purpose of this is twofold: to prepare the students for the types of "unfamiliar contexts" that can potentially arise in a Scholarship exam, and to prepare students for later tertiary study. This is particularly important as students who do well in Scholarship are sometimes offered the opportunity to skip first-year university courses, so such students are at a disadvantage if they have never seen things like matrices and have never worked beyond the scope of high school mathematics. Of course, the amount of extension material which you can cover with your group will be determined by the time available; you may find it difficult to do everything.

Each section in the text is followed by a small number of questions. It is vital that students at least attempt these questions, to cement what they are learning. The average Scholarship candidate may be unable to answer some questions, but should not be put off by this – because full worked answers are available, they should follow up every difficulty to best prepare themselves. Outstanding candidates should complete every exercise. While there are not a large number of questions on each topic, as compared with the average textbook, the questions develop quickly and cover every aspect included in the notes. If students find it difficult to grasp a particular concept, I would recommend searching for multitudes of basic questions in a standard Year 13 Calculus textbook or online.

You will notice that some notes in this booklet are boxed. This is extra material not included in the Student Workbook, offering extension that is best suited to candidates for the Outstanding Scholarship. These parts are usually algebraic derivations or manipulations that require skill and/or patience to reproduce. While they are not vital to the text, I believe these are important things to cover with any truly keen student, because the more they are exposed to, the more they can develop their mathematical skills and understanding and the more prepared they can be for unseen material in the examination and future studies.

If you have any feedback – criticisms or compliments – on the text, if you believe something should be added or expanded upon, if you have any questions, or if you find any errors, please don't hesitate to send me an e-mail at simon@scholcalc.co.nz or use the contact form online at www.scholcalc.co.nz.

Good luck to you and your students, and I hope that this package makes your preparation for Scholarship Calculus manageable, enjoyable and successful.

Simon Todd.

Contents – Year 12

Contents - Year 13

DIFFERENTIAL EQUATIONS

FORMING AND SOLVING SEPARABLE DIFFERENTIAL EQUATIONS

Often in questions, we are asked to form and solve differential equations, which can sometimes be quite difficult if it seems that we have insufficient or surplus information.

To solve a (first-order) differential equation specifically, we need at least two pieces of information: the equation itself, and an evaluation of the general solution to this at a given point (typically, the initial value). If the equation is given as a proportionality, i.e. contains an unevaluated constant of proportionality k, then we will require another evaluation of the solution to reach a full, specific answer which can be used to predict behaviour at any given point.

To form a differential equation, however, we really only need one piece of information: the style or order of change of the thing that is changing. We don't even need to know in what proportions it is changing, because as long as we have an evaluation of the system at two independent points, we can find this out later.

Say, for instance, that a spherical snowball is melting in such a way that its volume is decreasing at a rate proportional to its surface area.

We write $\frac{dV}{dt} = kA$ (where the constant of proportionality k will account for the deflation property).

We need something that links A and V. Obviously as we change one, the other changes – but what could we say causes these changes? The radius is also changing, and both A and V can be easily linked to r, so we pretend that the radius is changing in such a way that it causes volume and area to change in the relationship above. This allows us to transform A into a function of V and make our differential equation easy to integrate.

Now we know that $A = 4\pi r^2$, so $A \propto r^2$. We also know that $V = \frac{4}{3}\pi r^3$, so $V \propto r^3$.

We can therefore infer that $A \propto V^{2/3}$.

Because we work in proportionalities, the constants are irrelevant – they will be "absorbed" by k, and we can evaluate this k later.

Our equation thus becomes $\frac{dV}{dt} = kV^{2/3}$, which we can solve by separation of variables:

$$\int V^{-2/3}\, dV = \int k\, dt$$

$$\therefore 3V^{1/3} = kt + c$$

$$\therefore V = \left(\frac{kt + c}{3}\right)^3$$

Since k and c are arbitrary constants that we are yet to evaluate, they can "absorb" the division by 3, so that we get $V(t) = (kt + c)^3$.

(Note that we could NOT do this if k were not arbitrary, i.e. if we had been told to use k as a marker for some value rather than using it to represent proportionality.)

In fact, this is the general form to model a solution when a volume changes in proportion to a simply-related area such as cross-sectional or surface area, with the values of k and c changing depending on the situation.

Our two points of evaluation (which would be given in the question) can now be used to evaluate k and c.

If we knew, say, that the snowball began without having melted at all, and that it took 1 minute (= 60 seconds) to melt fully, we could evaluate these constants as follows:

$V(t) = (kt + c)^3$

$V(0) = c^3$

Marking full volume with 1, we would then have $1 = c^3 \Rightarrow c = 1$.

Then, assuming we measure t in seconds, making substitutions for the next evaluation point gives:

$V(60) = (60k + 1)^3 = 0$

$\therefore 60k + 1 = 0$

$\therefore k = \dfrac{-1}{60}$

Putting it all together gives our final equation modelling the melting of the snowball,

$V(t) = \left(\dfrac{-t}{60} + 1\right)^3 = \left(\dfrac{60 - t}{60}\right)^3$

(Note that, since the snowball cannot have negative volume and we cannot have negative time, $0 \leq t \leq 60$).

If we wanted to use this to find out when the snowball is $\frac{1}{8}$ of its maximum (initial) volume, say, we then substitute:

$\left(\dfrac{60 - t}{60}\right)^3 = \dfrac{1}{8}$

$\therefore \dfrac{60 - t}{60} = \dfrac{1}{2}$

$\therefore 60 - t = 30$

$\therefore t = 30$

So the snowball will be $\frac{1}{8}$ of its maximum (initial) volume after 30 seconds.

We could equivalently have done this example by related rates of change, which is typically a more complicated method as we end up solving the differential equation for a different variable than the one we want, and thus our constants become messy. However, this method can be necessary for things which are not simply-related (these typically yield equations unsolvable by separation).

Following this method, we would have realised $\dfrac{dV}{dt} = \dfrac{dV}{dr} \cdot \dfrac{dr}{dt} \Rightarrow \dfrac{dr}{dt} = \dfrac{dV}{dt} \bigg/ \dfrac{dV}{dr}$.

We know for a sphere, $V = \dfrac{4}{3}\pi r^3$ and $A = 4\pi r^2$.

Differentiating, we get $\frac{dV}{dr}=4\pi r^2$, so our differential equation becomes:

$$\frac{dr}{dt}=\frac{kA}{4\pi r^2}=k\cdot\frac{4\pi r^2}{4\pi r^2}=k$$

$$\therefore r=kt+c$$

We can substitute this back into our volume to get:

$$V(t)=\frac{4}{3}\pi(kt+c)^3$$

Since k and c are arbitrary constants that we are yet to evaluate, as before they can "absorb" the multiplication by $\frac{4}{3}\pi$ out the front, so that we get $V(t)=(kt+c)^3$.

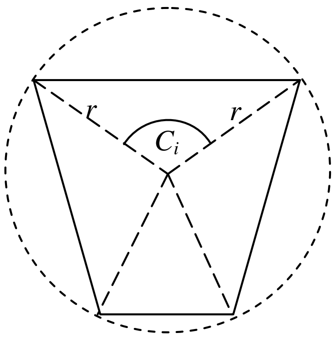
Thus we have the same answer as before.

Given, say, a filling or emptying tapered container, it is always easiest to consider the part that includes the point as the changing volume. That is, if a tapered container that sits upright (in the shape /\) is emptying, it is easier to consider the increasing emptiness (which is simply a smaller version of the container) than the decreasing fullness, and similarly if such a container is filling, it is simpler to consider the decreasing emptiness than the increasing fullness. If the container is in the shape \/, it is the changing fullness that is simplest to consider.

The moral is: always choose the simplest model, and don't forget that arbitrary constants can "absorb" any other constants and thus simplify your work incredibly.

In fact, we can apply this technique for any pyramid with a cyclic (not obligatorily regular) polygonal base whose volume is changing at a rate related to its instantaneous cross-sectional area.

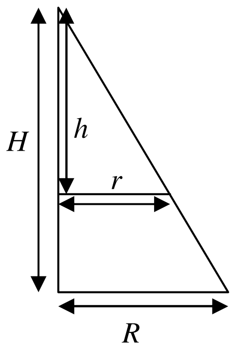
This is because any n-sided cyclic polygon inscribed in a circle of radius r is made up of n isosceles triangles (not necessarily all the same size) with equal sides r. If we let the top (i.e. central) angle for the ith triangle be C_i, then its area, given by the sine rule $A=\frac{1}{2}ab\sin C$, is $A_i=\frac{1}{2}r^2\sin C_i$ (as in the diagram below).



Clearly, the area of the polygonal base is the sum of all these triangular areas, which is proportional to r^2:

$$A=\sum_{i=1}^{n}A_i=\sum_{i=1}^{n}\frac{1}{2}r^2\sin C_i=r^2\sum_{i=1}^{n}\frac{\sin C_i}{2}\propto r^2$$

Now if we make a cut through the pyramid, from where the point is located through the centre of the inscribing circle on the base out to any base vertex, we can form a triangle as in the following diagram:



As can be seen, we form similar triangles between the triangle representing the height and radius of the whole pyramid (labelled with capitals H and R, which are constants) and the smaller one representing the instantaneous height, h, and radius, r.

We therefore have $\frac{h}{H}=\frac{r}{R}$, so $h=\frac{H}{R}r \propto r$ (height is proportional to radius at an instant).

(Note: in some cases, where the point is located beyond the base, such that the pyramid is on a significant lean, it would appear that such a right-angled triangle cannot be constructed, since the radius does not go all the way to the centre. In such cases, however, we have 2 lots of similar triangles – both involving a slope length – which still give the same relation.)

Since the volume of a pyramid with base area A and perpendicular height h is $V=\frac{1}{3}Ah$, we see $V \propto Ah$, and since we have found $A \propto r^2$ and $h \propto r$, we must have $V \propto r^3$.

Thus we infer $A \propto V^{2/3}$ for any pyramid with a cyclic base, and thus if $\frac{dV}{dt} \propto f(A)$, clearly $\frac{dV}{dt} \propto f\left(V^{2/3}\right)$.

As a side note, this makes perfect sense, since area is 2-dimensional and volume is 3-dimensional and since the cross-sectional shape of a pyramid is constant (and therefore area and volume are simply-related through their dimensions)

FORMING AND SOLVING SEPARABLE DIFFERENTIAL EQUATIONS EXERCISES

1. A conical icicle is placed on a barbeque, point down (V), and melts at a rate proportional to the area of its exposed cross-section on the hotplate (assume that this rate is unaffected by any cooling of the plate, and that the icicle does not tilt as it melts, so that its base is always parallel to the hotplate). If the icicle has melted to half its original volume after 20 seconds, how long will it take to melt completely?

2. The icicle in (1) is in fact the tip of a larger conical icicle which has been cut off. This remaining part of the icicle now has the shape of a frustrum, _/ , which is $\frac{2}{3}$ the volume of the original icicle (before being cut in two). After the other icicle has melted, this frustrum-shaped piece of ice is also put on the barbeque (which is now at a different temperature). However, it is more aerated than the other and melts at a rate proportional to the area of its exposed cross-section on the hotplate, squared. If this icicle takes 2 minutes to melt completely, when will its volume be $\frac{1}{4}$ that of the original (larger, conical) icicle?

3. Suppose $F=\frac{\pi a}{r^2\sqrt{b}}$ (where a and b are constants) and F is changing at a rate proportional to $G=\left(c+\sqrt{d}\right)\sqrt[5]{r}$ (where c and d are constants). If $F=F_0$ at time $t=0$ and $F=\frac{1024}{59049}F_0$ after 7613 minutes, how much longer will it take before $F=0$?

QUESTION TEN – DIFFERENTIAL EQUATIONS (9 marks)

a) Suppose c_1 and c_2 are constants.

(i) Show that $y(t) = c_1 e^{2t} + c_2 t e^{2t}$ is a solution to the differential equation $y'' - 4y' + 4y = 0$.

(ii) Hence determine the particular solution to $y'' - 4y' + 4y = 0$ if $y(1) = e^2$ and $y'(1) = 0$.

b) By using the substitution $z = \frac{y}{x}$, find the particular solution(s) to the differential equation

$\frac{dy}{dx} = \frac{y}{x} + \frac{x}{y\ln(x^2)}$, given that $x^2 > 1$ and $y(e^{1/2}) = 2$.

Give your answer in pure form, i.e. without decimals.

c) A candle has a cross-sectional shape of a regular octagon and tapers to a point, as shown below.

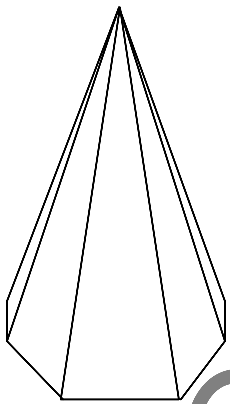


Fig 1: the tapered candle

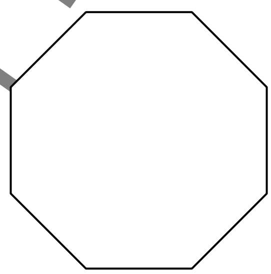


Fig 2: the candle's base

The candle burns (i.e. changes volume) at a rate which is inversely proportional to the exposed octagonal surface area of its top.

If the candle starts with $\frac{1}{10}$ of its volume already burnt down and it takes 9 hours for the candle to burn down completely from this starting point, when will the candle have burnt down to half its total volume?

(Assume the candle burns levelly and already-melted wax does not interfere.)

QUESTION TEN – DIFFERENTIAL EQUATIONS

a) Suppose c_1 and c_2 are constants.

(i) Show that $y(t) = c_1e^{2t} + c_2te^{2t}$ is a solution to the differential equation $y'' - 4y' + 4y = 0$.

By differentiation, we have:

$y(t) = c_1e^{2t} + c_2te^{2t}$

$y'(t) = 2c_1e^{2t} + c_2e^{2t} + 2c_2te^{2t}$

$y''(t) = 4c_1e^{2t} + 2c_2e^{2t} + 2c_2e^{2t} + 4c_2te^{2t}$

$= 4c_1e^{2t} + 4c_2e^{2t} + 4c_2te^{2t}$

Now, by substitution into the differential equation, we get:

$LHS = y'' - 4y' + 4y$

$= 4c_1e^{2t} + 4c_2e^{2t} + 4c_2te^{2t} - 4(2c_1e^{2t} + c_2e^{2t} + 2c_2te^{2t}) + 4(c_1e^{2t} + c_2te^{2t})$

$= 4c_1e^{2t} + 4c_2e^{2t} + 4c_2te^{2t} - 8c_1e^{2t} - 4c_2e^{2t} - 8c_2te^{2t} + 4c_1e^{2t} + 4c_2te^{2t}$

$= 0$

$= RHS$

Thus $y(t) = c_1e^{2t} + c_2te^{2t}$ is a solution to the differential equation $y'' - 4y' + 4y = 0$

(1 mark for answer.)

(ii) Hence determine the particular solution to $y'' - 4y' + 4y = 0$ if $y(1) = e^2$ and $y'(1) = 0$.

Our general solution, from part (i), is $y(t) = c_1e^{2t} + c_2te^{2t}$.

Now by substitution, $y(1) = c_1e^2 + c_2e^2 = e^2$.

$\therefore c_1 + c_2 = 1$

And $y'(1) = 2c_1e^2 + 3c_2e^2 = 0$.

$\therefore 2c_1 + 3c_2 = 0$

$\therefore c_2 = \frac{-2c_1}{3}$

Substituting this into our first equation gives: (1)

$c_1 - \frac{2c_1}{3} = 1$

$\therefore \frac{c_1}{3} = 1$

$\therefore c_1 = 3$

Substituting this back into the other equation gives $c_2 = -2$. (2)

Thus our particular solution is $y(t) = 3e^{2t} - 2te^{2t}$.

(1 mark for forming the simultaneous equations; 2 marks for the solution.)

b) By using the substitution $z=\frac{y}{x}$, find the particular solution(s) to the differential equation $\frac{dy}{dx}=\frac{y}{x}+\frac{x}{y\ln\left(x^2\right)}$, given that $x^2>1$ and $y\left(e^{1/2}\right)=2$. Give your answer in pure form, i.e. without decimals.

We make this substitution to remove the variable y, resulting in a separable differential equation in z and x.

Since $z=\frac{y}{x}\Rightarrow y=xz$.

We differentiate this implicitly with respect to x to find that $\frac{dy}{dx}=z+x\frac{dz}{dx}$.

Furthermore if $z=\frac{y}{x}$ then $\frac{x}{y}=\frac{1}{z}$, so that $\frac{x}{y\ln\left(x^2\right)}=\frac{1}{z\ln\left(x^2\right)}$.

Thus our differential equation $\frac{dy}{dx}=\frac{y}{x}+\frac{x}{y\ln\left(x^2\right)}$ becomes $z+x\frac{dz}{dx}=z+\frac{1}{z\ln\left(x^2\right)}$, which simplifies to

$x\frac{dz}{dx}=\frac{1}{z\ln\left(x^2\right)}$. (1)

We solve this by separation of variables:

$$\int z\,dz=\int\frac{1}{x\ln(x^2)}\,dx$$

Noting that the RHS integrand could also be written as $\frac{1}{2}\cdot\frac{2x}{x^2}\cdot\frac{1}{\ln\left(x^2\right)}$, we see that we can use the reverse chain rule on it. Thus we get:

$$\frac{1}{2}z^2=\frac{1}{2}\ln(\ln(x^2))+c$$

$\therefore z^2=\ln(\ln(x^2))+c$ (cancelling the halves is easier to work with; c is arbitrary so doesn't have to change.)

Note that we don't need absolute values with the logs because x^2 is always positive, and since $x^2>1$, $\ln(x^2)$ is always positive as well.

Substituting in our definition of z, this becomes $\left(\frac{y}{x}\right)^2=\ln\left(\ln\left(x^2\right)\right)+c$, i.e. $y^2=x^2(\ln(\ln(x^2))+c)$. (2)

Since $y\left(e^{1/2}\right)=2$, clearly we must have $\left[y\left(e^{1/2}\right)\right]^2=4$, so that by substitution we get:

$4=e(\ln(\ln(e))+c)$

$=e(\ln(1)+c)$

$=ec$

$\Rightarrow c=4e^{-1}$

Thus our particular solution is (implicitly) $y^2=x^2(\ln(\ln(x^2))+4e^{-1})$.

Square rooting gives $y=\pm x\sqrt{\ln(\ln(x^2))+4e^{-1}}$, but which one of these is the real solution?

We evaluate this using the boundary condition given:

$$y(e^{1/2}) = \pm e^{1/2}\sqrt{\ln(\ln(e)) + 4e^{-1}}$$

$$= \pm e^{1/2}\sqrt{\ln(1) + 4e^{-1}}$$

$$= \pm e^{1/2}\sqrt{4e^{-1}}$$

$$= \pm e^{1/2} \cdot 2e^{-1/2}$$

$$= \pm 2$$

Since the boundary condition was $y(e^{1/2}) = 2$, we clearly require the positive option.

Thus our solution is $y = x\sqrt{\ln(\ln(x^2)) + 4e^{-1}}$. (3)

(1 mark for altering the equation with the sub; 2 for solving for y^2;
3 for using boundary condition to find y and excluding the irrelevant solution.)

c) A candle has a cross-sectional shape of a regular octagon and tapers to a point, as shown below.

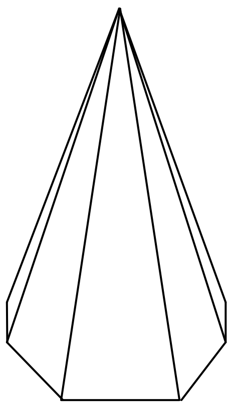


Fig 1: the tapered candle

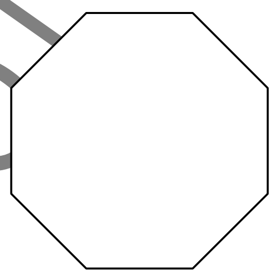


Fig 2: the candle's base

The candle burns (i.e. changes volume) at a rate which is inversely proportional to the exposed octagonal surface area of its top. If the candle starts with $\frac{1}{10}$ of its volume already burnt down and it takes 9 hours for the candle to burn down completely from this starting point, when will the candle have burnt down to half its total volume?

(Assume the candle burns levelly and already-melted wax does not interfere.)

If we considered the volume of wax, i.e. the decreasing candle volume, we would have to account for the volume of a truncated prism (due to the constantly-changing bit melting off the top), which is difficult. Our job becomes simpler if, instead, we consider the volume of "space", i.e. the increasing burnt candle, because then we need only consider the volume of the prism this represents.

From the information given, we have the volume of the candle changing at a rate which is inversely proportional to the surface area of its top. The volume of the "space" is thus also changing at a rate which is inversely proportional to surface area. We can write a differential equation for this:

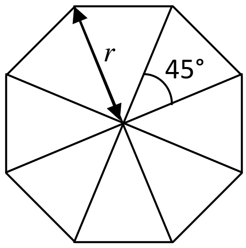
$$\frac{dV}{dt} = \frac{k}{A}$$

However, we cannot solve this without having A in terms of V.

The volume of a prism with regular cross-sectional area is given by $V=\frac{1}{3}A_b h$, where A_b is the area of the base. Since we are considering the increasing "space", the base of our prism is A, the surface area of the top.

How, though, do we link A and h?

Consider the "radius" of an octagon, pictured in the diagram below:

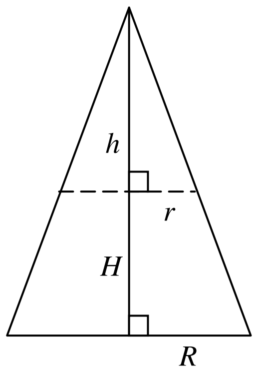


This radius splits the octagon up into 8 isosceles triangles which each have a top angle of 45° (from angles at a point).

By the sine rule for area, the area of each of these triangles is $\frac{1}{2}r^2\sin(45°)=\frac{r^2\sqrt{2}}{4}$

Thus the area of the octagon is $A=8\times\frac{r^2\sqrt{2}}{4}=2r^2\sqrt{2}$. (1)

Now consider a cross-section through the middle of the candle, along a diameter:



Let r be the instantaneous radius at any point, h be the instantaneous height, R be the radius of the base of the candle and H be the full height of the candle.

Then, from similar triangles, we see $\frac{h}{H}=\frac{r}{R}\Rightarrow h=\frac{H}{R}r$.

i.e. h and r are linearly related (r is a constant multiple of h).

Substituting these into our volume, we see that $V=\frac{2H\sqrt{2}}{3R}r^3$, so that $V\propto r^3$.

Thus, since $A\propto r^2$, we have $A\propto V^{2/3}$.

Since the proportionality will be absorbed by the (as yet undetermined) constant of proportionality, our differential equation becomes $\frac{dV}{dt}=\frac{k}{V^{2/3}}$, which we can solve by separation of variables: (2)

$$\int V^{2/3}\,dV=\int k\,dt$$

$$\therefore \frac{3}{5}V^{5/3}=kt+c$$

$$\therefore V^{5/3}=\frac{5(kt+c)}{3}$$

Note however that our constant of proportionality k and our constant of integration c are both unevaluated and arbitrary, so can 'absorb' the multiplication/division by other constants. Thus we get:

$$V^{5/3} = kt + c$$

$$\therefore V(t) = (kt + c)^{3/5}$$

For simplicity, let's assume the volume of the candle is 1. Remembering that our V represents the burnt "space", we have $V(0)=\frac{1}{10}$ and $V(9) = 1$, and we wish to find t such that $V(t)=1-\frac{1}{2}=\frac{1}{2}$.

We solve for the constants by substitution:

$$V(0) = (c)^{3/5} = \frac{1}{10} \Rightarrow c = \frac{1}{10^{5/3}} = \frac{\sqrt[3]{10}}{100}$$

$$V(9) = \left(9k + \frac{\sqrt[3]{10}}{100}\right)^{3/5} = 1 \Rightarrow k = \frac{1 - \frac{\sqrt[3]{10}}{100}}{9} = \frac{100 - \sqrt[3]{10}}{900}$$

Now we can solve for t:

$$V(t) = \left(\frac{100 - \sqrt[3]{10}}{900}t + \frac{\sqrt[3]{10}}{100}\right)^{3/5} = \left(\frac{\left(100 - \sqrt[3]{10}\right)t + 9\sqrt[3]{10}}{900}\right)^{3/5} = \frac{1}{2}$$

$$\therefore \frac{\left(100 - \sqrt[3]{10}\right)t + 9\sqrt[3]{10}}{900} = \frac{1}{2^{5/3}} = \frac{\sqrt[3]{2}}{4}$$

$$\therefore \left(100 - \sqrt[3]{10}\right)t + 9\sqrt[3]{10} = 225\sqrt[3]{2}$$

$$\therefore \left(100 - \sqrt[3]{10}\right)t = 225\sqrt[3]{2} - 9\sqrt[3]{10}$$

$$\therefore t = \frac{225\sqrt[3]{2} - 9\sqrt[3]{10}}{100 - \sqrt[3]{10}} = 2.699$$

Thus it will take 2.699 hours from its starting point for the candle to burn down to $\frac{1}{2}$ its total volume. (3)

(1 mark for finding the base area OR height expression;
2 for forming a separable DE;
3 for solving the DE using the boundary condition.)